

Quantized Feedback Stabilization of Sampled-Data Switched Linear Systems

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Abstract

We propose a stability analysis method for sampled-data switched linear systems with quantization. The available information to the controller is limited: the quantized state and switching signal at each sampling time. Switching between sampling times can produce the mismatch of the modes between the plant and the controller. Moreover, the coarseness of quantization makes the trajectory wander around, not approach, the origin. Hence the trajectory may leave the desired neighborhood if the mismatch leads to instability of the closed-loop system. For the stability of the switched systems, we develop a sufficient condition characterized by the *total mismatch time*. The relationship between the mismatch time and the dwell time of the switching signal is also discussed.

1 Introduction

In this paper, we consider a sampled-data switched linear system with a memoryless quantizer in Fig. 1. The available information to the controller is only the quantized state and switching signal at each sampling time. We then raise the questions: *What conditions are needed for the stability of the closed-loop system under such incomplete information? If the system is stable, how close can the trajectories get to the origin?*

Switched systems and quantized control have been studied extensively but separately; see, e.g., [Liberzon(2003b), Lin and Antsaklis(2009)] for switched systems and [Ishii and Francis(2002), Nair et al.(2007)] for quantized control. Few works examine the state behavior of a switched system with quantization and the effect of switching between sampling times. Recently, [Liberzon(2014)] has proposed an encoding and control strategy that achieves sampled-data quantized state feedback stabilization of switched systems. This strategy is rooted in the non-switched case in [Liberzon(2003a)]. In [Liberzon(2014)], the input of the controller is a discrete-valued and discrete-time signal, whereas the controller generates a continuous-valued and continuous-time output signal. In contrast, here we consider a controller whose *output as well as input* are *discrete-valued* and *discrete-time* signals.

[Ishii and Francis(2002), Ishii et al.(2004)] have studied the stability analysis of a sampled-data non-switched system of a memoryless quantizer. Since a memoryless quantizer does not give an accurate value of the state near the origin, asymptotic stability cannot be generally achieved. However, such a quantizer is useful because of the simplicity in implementation. [Ishii et al.(2004)] have developed a sufficient condition for a non-switched system to be quadratic attractive. The authors have also provided a randomized algorithm to verify this stability property in a computationally efficient way. In the present paper, we use this algorithm and a scheduling function with the revisitation property introduced in [Liberzon and Tempo(2004)]. The combined method constructs a common Lyapunov function guaranteeing the quadratic attractiveness of each subsystem.

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We face two challenges in the stability analysis of the switched system in Fig. 1. First, since only at sampling times we know which subsystem is active, we do not always use the feedback gain designed for the subsystem active at the present time. Therefore the closed-loop system may become unstable when switching occurs between sampling times. Second, after arriving at a certain neighborhood of the origin, the trajectory may not approach the origin anymore due to the coarseness of quantization. This implies that the trajectory can leave the desired neighborhood if switching makes the system unstable.

This paper is organized as follows. In Section 2, we state the switched system and the information structure together with basic assumptions. In Section 3, we first investigate the growth rate of the common Lyapunov function when switching occurs in a sampling interval. Next we develop a stability analysis method for the sampled-data switched system by using the *total mismatch time*, the total time when the modes mismatch between the plant and the controller. In Section 4, we briefly discuss the relationship between the mismatch time and the dwell time of the switching signal. Section 5 concludes this paper.

Notation

We denote by \mathbb{Z}_+ the set of non-negative integers $\{k \in \mathbb{Z} : k \geq 0\}$. For a set $\Omega \subset \mathbb{R}^n$, $\text{Cl}(\Omega)$, $\text{Int}(\Omega)$, and $\partial\Omega$ are its closure, interior, and boundary, respectively.

Let M^\top denote the transpose of $M \in \mathbb{R}^{n \times m}$. The Euclidean norm of $v \in \mathbb{R}^n$ is defined by $\|v\| = (v^\top v)^{1/2}$. For $M \in \mathbb{R}^{m \times n}$, its Euclidean induced norm is defined by $\|M\| = \sup\{\|Mv\| : v \in \mathbb{R}^n, \|v\| = 1\}$ and equals the largest singular value of M . Let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the largest and the smallest eigenvalue of $P \in \mathbb{R}^{n \times n}$.

Let T_s be a sampling period. For $t \geq 0$, we define $[t]^-$ by

$$[t]^- = kT_s \quad \text{if} \quad kT_s \leq t < (k+1)T_s \quad (k \in \mathbb{Z}_+).$$

2 Sampled-data Switched Systems with Quantization

2.1 Switched systems

Consider the continuous-time switched linear system

$$\dot{x} = A_\sigma x + B_\sigma u, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input. For a finite index set \mathcal{P} , the mapping $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is right-continuous and piecewise constant. We call σ *switching signal* and the discontinuities of σ *switching times*.

We assume that all subsystems are stabilizable and that only finitely many switches occurs on a finite interval:

Assumption 2.1. *For every $p \in \mathcal{P}$, (A_p, B_p) is stabilizable, i.e., there exists $K_p \in \mathbb{R}^{m \times n}$ such that $A_p + B_p K_p$ is Hurwitz. Furthermore, σ has finitely many switching times on every finite interval.*

Figure 1: Sampled-data switched system with quantization, where T_s is a sampling period

2.2 Quantized sampled-data system

Let $T_s > 0$ be a sampling period. The sampler S_{T_s} is given by

$$S_{T_s} : (x, \sigma) \mapsto (x(kT_s), \sigma(kT_s)) \quad (k \in \mathbb{Z}_+)$$

and the zero-th hold H_{T_s} by

$$H_{T_s} : u_d \mapsto u(t) = u_d(k), \quad t \in [kT_s, (k+1)T_s) \quad (k \in \mathbb{Z}_+).$$

We now state the definition of a memoryless quantizer Q given in [Ishii et al.(2004)]. For an index set \mathcal{S} , the partition $\{\mathcal{Q}_j\}_{j \in \mathcal{S}}$ of \mathbb{R}^n is said to be *finite* if for every bounded set B , there exists a finite subset \mathcal{S}_f of \mathcal{S} such that $B \subset \bigcup_{j \in \mathcal{S}_f} \mathcal{Q}_j$. We define the quantizer Q with respect to the finite partition $\{\mathcal{Q}_j\}_{j \in \mathcal{S}_f}$ by

$$\begin{aligned} Q : \mathbb{R}^n &\rightarrow \{q_j\}_{j \in \mathcal{S}_f} \subset \mathbb{R}^n \\ x &\mapsto q_j \quad \text{if } x \in \mathcal{Q}_j \quad (j \in \mathcal{S}_f). \end{aligned}$$

The second assumption is that $Q(x) = 0$ if x is close to the origin.

Assumption 2.2. If $\text{Cl}(\mathcal{Q}_j)$ contains the origin, then $q_j = 0$.

Let q_x be the output of the zero-th hold whose input is the quantized state at sampling times, i.e., $q_x(t) = Q(x([t]^-))$. Note that in Fig. 1, the control input u is given by

$$u(t) = K_{\sigma([t]^-)} q_x(t). \quad (2.2)$$

Let $P \in \mathbb{R}^{n \times n}$ be positive-definite and define the quadratic Lyapunov function $V(x) = x^\top P x$ for $x \in \mathbb{R}^n$. Its time derivative \dot{V} along the trajectory of (2.1) with (2.2) is given by

$$\dot{V}(x(t), q_x(t), \sigma(t)) = (A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma([t]^-)}q_x(t))^\top P x(t) + x(t)^\top P (A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma([t]^-)}q_x(t)) \quad (2.3)$$

if t is not a switching time nor a sampling time.

For $p, q \in \mathcal{P}$ with $p \neq q$, we also define \dot{V}_p and $\dot{V}_{p,q}$ by

$$\dot{V}_p(x(t), q_x(t)) = (A_p x(t) + B_p K_p q_x(t))^\top P x(t) + x(t)^\top P (A_p x(t) + B_p K_p q_x(t)) \quad (2.4)$$

$$\dot{V}_{p,q}(x(t), q_x(t)) = (A_p x(t) + B_p K_q q_x(t))^\top P x(t) + x(t)^\top P (A_p x(t) + B_p K_q q_x(t)). \quad (2.5)$$

Then \dot{V}_p and $\dot{V}_{p,q}$ are the time derivatives of V along the trajectories of the systems $(A_p, B_p K_p)$ and $(A_p, B_p K_q)$, respectively.

We assume that a common Lyapunov function guarantees the *quadratic attractiveness* of every individual mode.

Assumption 2.3. Consider the following sampled-data non-switched systems with quantization:

$$\dot{x} = A_p x + B_p u, \quad u = K_p q_x \quad (p \in \mathcal{P}). \quad (2.6)$$

Let C be a positive number and suppose that R and r satisfy $R > r > 0$. Then there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ such that for all $p \in \mathcal{P}$, every trajectory x of the system (2.6) with $x(0) \in \bar{\mathcal{E}}_P(R)$ satisfies

$$\dot{V}_p(x(t), q_x(t)) \leq -C \|x(t)\|^2 \quad (2.7)$$

or $x(t) \in \underline{\mathcal{E}}_P(r)$ for $t \geq 0$, where $\bar{\mathcal{E}}_P(R)$ and $\underline{\mathcal{E}}_P(r)$ are given by

$$\begin{aligned} \bar{\mathcal{E}}_P(R) &= \{x \in \mathbb{R}^n : V(x) \leq R^2 \lambda_{\max}(P)\} \\ \underline{\mathcal{E}}_P(r) &= \{x \in \mathbb{R}^n : V(x) \leq r^2 \lambda_{\min}(P)\}. \end{aligned}$$

Assumption 2.3 implies the followings: If we have no switches, then the common Lyapunov function V decreases at a certain rate until $V \leq r^2 \lambda_{\min}(P)$. Furthermore, $\underline{\mathcal{E}}_P(r)$ as well as $\overline{\mathcal{E}}_P(R)$ are invariant sets.

The objective of the present paper is to find switching conditions for the switched system in Fig. 1 to be quadratic attractive. We also determine how small a neighborhood the trajectories arrive at and remain in.

Remark 2.4. (a) In the non-sampled case, the existence of common Lyapunov functions is a sufficient condition for stability under arbitrary switching; see, e.g., [Liberzon(2003b), Lin and Antsaklis(2009)]. For sampled-data switched systems, however, such functions do not guarantee the stability because switching between sampling times may make the closed-loop system unstable.

(b) For plants with a single mode, [Ishii et al.(2004)] proposed a randomized algorithm for the computation of P in Assumption 2.3. Combining the algorithm with a scheduling function that has the revisitation property in [Liberzon and Tempo(2004)], we can efficiently compute the desired common Lyapunov function. Since this is an immediate consequence of the above two works, we omit the details.

3 Stabilization with Limited Information

3.1 Upper bounds of $\dot{V}_{p,q}$

Assumption 2.3 gives an upper bound (2.7) of \dot{V}_p , i.e., \dot{V} when we use the feedback gain designed for the currently active subsystem. In this subsection, we will find an upper bound of $\dot{V}_{p,q}$, i.e., \dot{V} when intersample switching leads to the mismatch of the modes between the plant and the feedback gain. To this end, we investigate the state behavior in sampling intervals.

Let us first examine the relationship among the original state $x(t)$, the sampled state $x([t]^-)$, and the sampled quantized state $q_x(t)$.

The partition $\{\mathcal{Q}_j\}_{j \in \mathcal{S}_f}$ is finite. Moreover, Assumption 2.2 shows that if $x_k \rightarrow 0$ ($k \rightarrow \infty$) for some sequence $\{x_k\} \subset \mathcal{Q}_j$, then $Q(x) = 0$ for all $x \in \mathcal{Q}_j$. Hence there exists $\alpha_0 > 0$ such that

$$\|B_p K_q Q(x)\| \leq \alpha_0 \|x\|, \quad (3.1)$$

for $p, q \in \mathcal{P}$ and $x \in \overline{\mathcal{E}}_P(R)$. We also define Λ by

$$\Lambda = \max_{p \in \mathcal{P}} \|A_p\|.$$

The next result gives an upper bound on the norm of the sampled state $x([t]^-)$ with the original state $x(t)$.

Lemma 3.1. *Consider the switched system (2.1) with (2.2), where σ has finitely many switching times on every finite interval. Suppose that*

$$\eta := \alpha_0 \frac{e^{\Lambda T_s} - 1}{\Lambda} < 1, \quad (3.2)$$

and define α_1 by

$$\alpha_1 = \frac{e^{\Lambda T_s}}{1 - \eta}.$$

Then we have

$$\|x([t]^-)\| < \alpha_1 \|x(t)\| \quad (3.3)$$

for all $t \geq 0$ with $x([t]^-) \in \overline{\mathcal{E}}_P(R)$.

Proof. It suffices to prove (3.3) for $x(0) \in \bar{\mathcal{E}}_P(R)$ and $t \in [0, T_s)$.

Let $\Phi(\tau_1, \tau_2)$ denote the state-transition matrix of (2.1) for $\tau_1 \geq \tau_2$. If switching does not occur, $\Phi(\tau_1, \tau_2)$ is given by $\Phi(\tau_1, \tau_2) = e^{(\tau_1 - \tau_2)A_{\sigma(0)}}$. If t_1, t_2, \dots, t_m are switching times on an interval $[\tau_2, \tau_1)$, then we have

$$\Phi(\tau_1, \tau_2) = e^{(\tau_1 - t_m)A_{\sigma(t_m)}} \prod_{k=1}^{m-1} e^{(t_{k+1} - t_k)A_{\sigma(t_k)}} \cdot e^{(t_1 - \tau_2)A_{\sigma(\tau_2)}}.$$

Since

$$x(t) = \Phi(t, 0)x(0) + \int_0^t \Phi(t, \tau)B_{\sigma(\tau)}K_{\sigma(0)}q_x(\tau)d\tau \quad (3.4)$$

and since $\Phi(\tau, 0)^{-1} = \Phi(t, 0)^{-1}\Phi(t, \tau)$, it follows that

$$x(0) = \Phi(t, 0)^{-1}x(t) + \int_0^t \Phi(\tau, 0)^{-1}B_{\sigma(\tau)}K_{\sigma(0)}q_x(\tau)d\tau.$$

This leads to

$$\|x(0)\| \leq \|\Phi(t, 0)^{-1}\| \cdot \|x(t)\| + \left\| \int_0^t \Phi(\tau, 0)^{-1}B_{\sigma(\tau)}K_{\sigma(0)}q_x(\tau)d\tau \right\|. \quad (3.5)$$

Let t_1, t_2, \dots, t_m be switching times on the interval $[0, t)$. Since $\|e^{\tau A}\| \leq e^{\tau\|A\|}$ for $\tau \geq 0$, we obtain

$$\begin{aligned} \|\Phi(t, 0)^{-1}\| &\leq e^{t_1\|A_{\sigma(0)}\|} \cdot \prod_{k=1}^{m-1} e^{(t_{k+1} - t_k)\|A_{\sigma(t_k)}\|} \cdot e^{(t - t_m)\|A_{\sigma(t_m)}\|} \\ &\leq e^{\Lambda t} < e^{\Lambda T_s}. \end{aligned} \quad (3.6)$$

It is obvious that the equation above holds in the non-switched case. Since $q_x(\tau) = q_x(0)$ when $0 \leq \tau \leq t$ ($< T_s$), it follows from (3.1) that

$$\begin{aligned} \left\| \int_0^t \Phi(\tau, 0)^{-1}B_{\sigma(\tau)}K_{\sigma(0)}q_x(\tau)d\tau \right\| &\leq \int_0^t \|\Phi(\tau, 0)^{-1}\| \cdot \|B_{\sigma(\tau)}K_{\sigma(0)}q_x(\tau)\|d\tau \\ &\leq \alpha_0 \int_0^t e^{\Lambda\tau}d\tau \|x(0)\| \\ &\leq \alpha_0 \frac{e^{\Lambda T_s} - 1}{\Lambda} \|x(0)\| = \eta \|x(0)\|. \end{aligned} \quad (3.7)$$

Substituting (3.6) and (3.7) into (3.5), we obtain

$$\|x(0)\| < e^{\Lambda T_s} \|x(t)\| + \eta \|x(0)\|.$$

Thus if (3.2) holds, we derive (3.3). □

Let us next develop an upper bound on the norm of the error $x(t) - x([t]^-)$ due to sampling. To this end, we show the following proposition:

Proposition 3.2. *Let $\Phi(t, 0)$ be the state-transition map of (2.1) as above. Then*

$$\|\Phi(t, 0) - I\| \leq e^{\Lambda t} - 1. \quad (3.8)$$

Proof. Let us first show the case without switching, i.e.,

$$\|e^{tA_{\sigma(0)}} - I\| \leq e^{\Lambda t} - 1. \quad (3.9)$$

Define the partial sum S_N of $e^{tA_{\sigma(0)}} - I$ by

$$S_N(t) = \sum_{k=0}^N \frac{1}{k!} (tA_{\sigma(0)})^k - I = \sum_{k=1}^N \frac{1}{k!} (tA_{\sigma(0)})^k$$

Then for $t \geq 0$

$$\begin{aligned} \|S_N(t)\| &\leq \sum_{k=1}^N \frac{1}{k!} (t\|A_{\sigma(0)}\|)^k \\ &= \sum_{k=0}^N \frac{1}{k!} (t\|A_{\sigma(0)}\|)^k - 1 \\ &\leq \sum_{k=0}^{\infty} \frac{1}{k!} (t\|A_{\sigma(0)}\|)^k - 1 \\ &= e^{t\|A_{\sigma(0)}\|} - 1 \leq e^{\Lambda t} - 1. \end{aligned}$$

If we let $N \rightarrow \infty$, we obtain (3.9).

We now prove (3.8) in the switched case. Let t_1, t_2, \dots, t_m be the switching times in the interval $(0, t)$. Let $t_0 = 0$ and $t_{m+1} = t$. Then (3.8) is equivalent to

$$\left\| \prod_{l=0}^m e^{(t_{l+1}-t_l)A_{\sigma(t_l)}} - I \right\| \leq e^{\Lambda t} - 1. \quad (3.10)$$

We have already shown the case $m = 0$, i.e., non-switched case. The general case follows by induction. For $m \geq 1$,

$$\begin{aligned} \left\| \prod_{l=0}^m e^{(t_{l+1}-t_l)A_{\sigma(t_l)}} - I \right\| &\leq \left\| e^{(t_{m+1}-t_m)A_{\sigma(t_m)}} \left(\prod_{l=0}^{m-1} e^{(t_{l+1}-t_l)A_{\sigma(t_l)}} - I \right) \right\| + \|e^{(t_{m+1}-t_m)A_{\sigma(t_m)}} - I\| \\ &\leq \|e^{(t_{m+1}-t_m)A_{\sigma(t_m)}}\| \cdot \left\| \prod_{l=0}^{m-1} e^{(t_{l+1}-t_l)A_{\sigma(t_l)}} - I \right\| + \|e^{(t_{m+1}-t_m)A_{\sigma(t_m)}} - I\|. \end{aligned}$$

Hence if (3.10) holds with $m-1$ in place of m , then

$$\begin{aligned} &\|e^{(t-t_{k+1})A_{\sigma(t_{k+1})}}\| \cdot \left\| \prod_{l=0}^k e^{(t_{l+1}-t_l)A_{\sigma(t_l)}} - I \right\| + \|e^{(t-t_{k+1})A_{\sigma(t_{k+1})}} - I\| \\ &\leq e^{\Lambda(t-t_{k+1})} (e^{\Lambda t_{k+1}} - 1) + (e^{\Lambda(t-t_{k+1})} - 1) \\ &= e^{\Lambda t} - 1. \end{aligned}$$

Thus we obtain (3.10) with $m = k+1$. □

Lemma 3.3. Consider the switched system (2.1) with (2.2), where σ has finitely many switching times on every finite interval. Define β_1 by

$$\beta_1 = (e^{\Lambda T_s} - 1) \left(1 + \frac{\alpha_0}{\Lambda} \right)$$

Then we have

$$\|x(t) - x([t]^-)\| < \beta_1 \|x([t]^-)\| \quad (3.11)$$

for all $t \geq 0$ with $x([t]^-) \in \bar{\mathcal{E}}_P(R)$.

Proof. As in the proof of Lemma 3.1, it suffices to prove (3.11) for all $x(0) \in \overline{\mathcal{E}}_P(R)$ and $t \in [0, T_s]$. By (3.4), we obtain

$$x(t) - x(0) = (\Phi(t, 0) - I)x(0) + \int_0^t \Phi(t, \tau) B_{\sigma(\tau)} K_{\sigma(0)} q_x(\tau) d\tau.$$

This leads to

$$\|x(t) - x(0)\| \leq \|\Phi(t, 0) - I\| \cdot \|x(0)\| + \left\| \int_0^t \Phi(t, \tau) B_{\sigma(\tau)} K_{\sigma(0)} q_x(\tau) d\tau \right\|. \quad (3.12)$$

Proposition 3.2 provides the following upper bound of the first term of the right side of (3.12):

$$\|\Phi(t, 0) - I\| \leq e^{\Lambda t} - 1 < e^{\Lambda T_s} - 1.$$

Since a calculation similar to (3.6) shows that $\|\Phi(t, \tau)\| \leq e^{\Lambda(t-\tau)}$. Hence as in (3.7),

$$\left\| \int_0^t \Phi(t, \tau) B_{\sigma(\tau)} K_{\sigma(0)} q_x(\tau) d\tau \right\| \leq \alpha_0 \frac{e^{\Lambda T_s} - 1}{\Lambda} \|x(0)\|, \quad (3.13)$$

we obtain (3.11) by combining (3.8) with (3.13). \square

Similarly to (3.1), to each $p, q \in \mathcal{P}$ with $p \neq q$, there correspond positive numbers $\alpha_2(p, q)$ and $\beta_2(p, q)$ such that

$$\|PB_p(K_q - K_p)Q(x)\| \leq \alpha_2(p, q)\|x\| \quad (3.14)$$

$$\|PB_p K_q(Q(x) - x)\| \leq \beta_2(p, q)\|x\| \quad (3.15)$$

for $x \in \overline{\mathcal{E}}_P(R)$.

Finally we derive upper bounds on the norm of the sampled quantized state $q_x(t)$ and that of the error $q_x(t) - x(t)$ due to sampling and quantization from the original state $x(t)$.

Theorem 3.4. Consider the switched system (2.1) with (2.2), where σ has finitely many switching times on every finite interval. Define α_1 and β_1 as in Lemmas 3.1 and 3.3. If $\alpha(p, q)$ and $\beta(p, q)$ are defined by

$$\begin{aligned} \alpha(p, q) &= \alpha_1 \alpha_2(p, q) \\ \beta(p, q) &= \alpha_1 (\beta_1 \|PB_p K_q\| + \beta_2(p, q)), \end{aligned}$$

then $\alpha(p, q)$ and $\beta(p, q)$ satisfy

$$\|PB_p(K_q - K_p)q_x(t)\| < \alpha(p, q)\|x(t)\| \quad (3.16)$$

$$\|PB_p K_q(q_x(t) - x(t))\| < \beta(p, q)\|x(t)\| \quad (3.17)$$

for all $t \geq 0$ with $x([t]^-) \in \overline{\mathcal{E}}_P(R)$.

Proof. We obtain the first inequality (3.16) by (3.3) and (3.14). Also (3.11) and (3.15) show that

$$\begin{aligned} \|PB_p K_q(q_x(t) - x(t))\| &\leq \|PB_p K_q(q_x(t) - x([t]^-))\| + \|PB_p K_q\| \cdot \|x([t]^-) - x(t)\| \\ &< (\beta_1 \|PB_p K_q\| + \beta_2(p, q)) \|x([t]^-)\| \\ &< \alpha_1 (\beta_1 \|PB_p K_q\| + \beta_2(p, q)) \|x(t)\|. \end{aligned}$$

Thus (3.17) holds. \blacksquare \square

An upper bound on $\dot{V}_{p,q}$ can be obtained as follows.

First, since $V_{p,q}$ satisfies

$$\dot{V}_{p,q}(x(t), q_x(t)) = \dot{V}_p(x(t), q_x(t)) + 2x(t)^\top PB_p(K_q - K_p)q_x(t),$$

it follows from (2.7) and Theorem 3.4 that

$$\dot{V}_{p,q}(x(t), q_x(t)) \leq (2\alpha(p, q) - C)\|x(t)\|^2. \quad (3.18)$$

for all $t \geq 0$ with $x(t) \in \bar{\mathcal{E}}_P(R) \setminus \underline{\mathcal{E}}_P(r)$. Here we used

$$\{x(t) : t \geq 0\} \supset \{x([t]^-) : t \geq 0\}. \quad (3.19)$$

The first bound (3.18) can be negative if the sampling period T_s and the gain difference $\|K_p - K_q\|$ are sufficiently small.

Second, since $\dot{V}_{p,q}$ also satisfies

$$\dot{V}_{p,q}(x(t), q_x(t)) = 2x(t)^\top P(A_p + B_p K_q)x(t) + 2x(t)^\top PB_p K_q(q_x(t) - x(t)),$$

we see from (3.17) that

$$\dot{V}_{p,q}(x(t), q_x(t)) \leq 2(\|P(A_p + B_p K_q)\| + \beta(p, q))\|x(t)\|^2 \quad (3.20)$$

for all $t \geq 0$ with $x([t]^-) \in \bar{\mathcal{E}}_P(R)$. Fast sampling and fine quantization make the second bound (3.20) small.

Define D by

$$D = \max_{p \neq q} \min \{2\alpha(p, q) - C, 2(\|P(A_p + B_p K_q)\| + \beta(p, q))\}. \quad (3.21)$$

Using (3.19) again, we obtain

$$\dot{V}_{p,q}(x(t), q_x(t)) \leq D\|x(t)\|^2 \quad (3.22)$$

for $p, q \in \mathcal{P}$ with $p \neq q$ and for $t \geq 0$ with $x(t) \in \bar{\mathcal{E}}_P(R) \setminus \underline{\mathcal{E}}_P(r)$,

We assume that D in (3.21) satisfies $D \geq 0$. This assumption involves no loss of generality. In fact, if $D < 0$, then \dot{V} in (2.3) is negative for all σ . This implies that every trajectory with its initial state in $\bar{\mathcal{E}}_P(R)$ goes into $\underline{\mathcal{E}}_P(r)$ and remains there for all switching signals. Hence the stabilization of the system (2.1) with (2.2) can be achieved without any information about switching signals. Thus the problem to be posed is trivial.

3.2 Stability analysis with total mismatch time

We are now in position to analyze the stability of the switched system (2.1) with (2.2) by the two upper bounds (2.7) and (3.22) of \dot{V} . Note that the former bound (2.7) is for the case $\sigma(t) = \sigma([t]^-)$, while the latter (3.22) for the case $\sigma(t) \neq \sigma([t]^-)$.

Definition 3.5. For $\tau_1 > \tau_2 \geq 0$, we define the total mismatch time $\mu(\tau_1, \tau_2)$ by

$$\mu(\tau_1, \tau_2) = \text{the length of } \{\tau \in [\tau_2, \tau_1) : \sigma(\tau) \neq \sigma([\tau]^-)\} \quad (3.23)$$

More explicitly, the length of an interval means its Lebesgue measure. We shall not, however, use any measure theory because σ has only finitely many discontinuities.

Define C_P and D_P by

$$C_P = \frac{C}{\lambda_{\max}(P)}, \quad D_P = \frac{D}{\lambda_{\min}(P)}.$$

First we discuss the state behavior when it is outside of $\underline{\mathcal{E}}_P(r)$. The following lemma suggests that every trajectory with its initial state in $\text{Int}(\bar{\mathcal{E}}_P(R))$ goes into $\underline{\mathcal{E}}_P(r)$ if μ is sufficiently small.

Lemma 3.6. *Let Assumptions 2.1, 2.2, and 2.3 hold, and let $L \geq 0$ satisfy*

$$L < \frac{C_P}{C_P + D_P}. \quad (3.24)$$

If $\mu(t, 0)$ achieves

$$\mu(t, 0) \leq Lt \quad (3.25)$$

for $t > 0$, then there exists $T_r \geq 0$ such that for every $x(0) \in \text{Int}(\bar{\mathcal{E}}_P(R))$ and $\sigma(0) \in \mathcal{P}$, $x(T_r) \in \underline{\mathcal{E}}_P(r)$ and $x(t) \in \text{Int}(\bar{\mathcal{E}}_P(R))$ for all $t \in [0, T_r]$.

Proof. First we show that the trajectory x does not leave $\text{Int}(\bar{\mathcal{E}}_P(R))$ without belonging to $\underline{\mathcal{E}}_P(r)$. That is, there does not exist $T_R > 0$ such that

$$x(T_R) \in \partial\bar{\mathcal{E}}_P(R) \quad (3.26)$$

$$x(t) \in \text{Int}(\bar{\mathcal{E}}_P(R)) \setminus \underline{\mathcal{E}}_P(r) \quad (0 \leq t < T_R). \quad (3.27)$$

Assume, to reach a contradiction, (3.26) and (3.27) hold for some $T_R > 0$. Recall that

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) = x^\top Px \leq \lambda_{\max}(P)\|x\|^2$$

for $x \in \mathbb{R}^n$. It follows from (2.7) and (3.22) that

$$\begin{aligned} \dot{V}_p(x(t), q_x(t)) &\leq -C_P V(x(t)) \\ \dot{V}_{p,q}(x(t), q_x(t)) &\leq D_P V(x(t)). \end{aligned} \quad (3.28)$$

By (3.28), a successive calculation at each switching time shows that

$$V(x(T_R)) \leq \exp(D_P \mu(T_R, 0) - C_P(T_R - \mu(T_R, 0))) V(x(0)). \quad (3.29)$$

Since (3.25) gives

$$D_P \mu(t, 0) - C_P(t - \mu(t, 0)) \leq ((C_P + D_P)L - C_P)t \quad (3.30)$$

for $t > 0$, it follows from (3.24) and $x(0) \in \text{Int}(\bar{\mathcal{E}}_P(R))$ that

$$V(x(T_R)) < V(x(0)) < R^2 \lambda_{\max}(P).$$

However, (3.26) shows that $V(x(T_R)) = R^2 \lambda_{\max}(P)$, and we have a contradiction.

Let us next prove that $x(T_r) \in \underline{\mathcal{E}}_P(r)$ for some $T_r \geq 0$.

Suppose $x(t) \notin \underline{\mathcal{E}}_P(r)$ for all $t \geq 0$. Then since the discussion above shows that $x(t) \in \text{Int}(\bar{\mathcal{E}}_P(R)) \setminus \underline{\mathcal{E}}_P(r)$ for $t \geq 0$, we obtain (3.29) with arbitrary $t \geq 0$ in place of T_R . Hence (3.24) and (3.30) show that $V(x(t)) \rightarrow 0$ as $t \rightarrow \infty$. However this contradicts $x(t) \notin \underline{\mathcal{E}}_P(r)$, i.e., $V(x(t)) > r^2 \lambda_{\min}(P)$. Thus there exists $T_r \geq 0$ such that $x(T_r) \in \underline{\mathcal{E}}_P(r)$. \square

From the next result, we see that the trajectory leaves $\underline{\mathcal{E}}_P(r)$ only if switching occurs between sampling times.

Lemma 3.7. *Let Assumptions 2.1, 2.2, and 2.3 hold. Let the trajectory $x(t)$ leave $\underline{\mathcal{E}}_P(r)$ when $t = T_0$. More precisely, there exists $\delta > 0$ such that*

$$x(T_0) \in \partial\underline{\mathcal{E}}_P(r), \quad x(T_0 + \varepsilon) \notin \underline{\mathcal{E}}_P(r) \quad (0 < \varepsilon < \delta). \quad (3.31)$$

Then $\sigma(T_0) \neq \sigma([T_0]^-)$.

Proof. Suppose $\sigma(T_0) = \sigma([T_0]^-)$. Assume that $\sigma(T) \neq \sigma([T]^-)$ for some $T > T_0$. Let T_1 be the smallest number of such T . Define an interval I_δ by

$$I_\delta = (0, \min\{\delta, T_1 - T_0\}).$$

If there does not exist $T > T_0$ with $\sigma(T) \neq \sigma([T]^-)$, then we define I_δ by $I_\delta = (0, \delta)$. Since $\sigma(T_0 + \varepsilon) = \sigma([T_0 + \varepsilon]^-)$ for $\varepsilon \in I_\delta$, it follows from (2.7) that

$$\dot{V}(x(T_0 + \varepsilon)) \leq -C\|x((T_0 + \varepsilon))\|^2 \leq 0 \quad (\varepsilon \in I_\delta).$$

However, since (3.31) gives $x(T_0 + \varepsilon) \notin \underline{\mathcal{E}}_P(r)$,

$$V(x(T_0 + \varepsilon)) > r^2 \lambda_{\min}(P) = V(x(T_0)) \quad (\varepsilon \in I_\delta).$$

Hence we have a contradiction by the mean value theorem. Thus $\sigma(T_0) \neq \sigma([T_0]^-)$. \square

The next result shows that if the trajectory enters into $\underline{\mathcal{E}}_P(r)$, it keeps roaming a little larger ellipsoid than $\underline{\mathcal{E}}_P(r)$.

Lemma 3.8. *Let Assumptions 2.1, 2.2, and 2.3 hold. Suppose that there exist $T_0 \geq 0$ and $\delta > 0$ satisfying (3.31). Let $a > 1$ satisfy*

$$a^2 r^2 \lambda_{\min}(P) < R^2 \lambda_{\max}(P) \quad (3.32)$$

and define $b(a)$ by

$$b(a) = \frac{2 \log a}{C_P + D_P}. \quad (3.33)$$

Pick $L \geq 0$ with (3.24) and suppose that $\mu(t, T_0)$ satisfies

$$\mu(t, T_0) \leq b(a) + L(t - T_0) \quad (3.34)$$

for all $t > T_0$. Then there exists $T_1 > T_0$ such that for every $\sigma(T_0) \in \mathcal{P}$, $x(T_1) \in \underline{\mathcal{E}}_P(r)$ and $x(t) \in \text{Int}(\underline{\mathcal{E}}_P(ar))$ for $t \in [T_0, T_1]$.

Proof. By (3.31), $x(t) \notin \underline{\mathcal{E}}_P(r)$ for $t \in (T_0, T_0 + \delta)$. Moreover, (3.34) shows that as long as $x(t) \in \overline{\mathcal{E}}_P(R) \setminus \underline{\mathcal{E}}_P(r)$, $V(x(t))$ satisfies

$$V(x(t)) \leq \exp((C_P + D_P)L - C_P)(t - T_0)) \exp((C_P + D_P)b(a))V(x(T_0)). \quad (3.35)$$

On the other hand, since $x(T_0) \in \partial \underline{\mathcal{E}}_P(r)$, it follows from (3.33) that

$$\exp((C_P + D_P)b(a))V(x(T_0)) = a^2 r^2 \lambda_{\min}(P). \quad (3.36)$$

Note that (3.32) is equivalent to

$$\underline{\mathcal{E}}_P(ar) = \{x \in \mathbb{R}^n : V(x) \leq a^2 r^2 \lambda_{\min}(P)\} \subset \overline{\mathcal{E}}_P(R).$$

As in the proof of Lemma 3.6, we see from (3.35) that $x(T_1) \in \underline{\mathcal{E}}_P(r)$ for some $T_1 > T_0$. Substituting (3.36) into (3.35), we also obtain $V(x(t)) < a^2 r^2 \lambda_{\min}(P)$. Thus $x(t) \in \text{Int}(\underline{\mathcal{E}}_P(ar))$ for $t \in [T_0, T_1]$. \square

Referring to Lemmas 3.6, 3.7, and 3.8, we immediately derive the following result:

Theorem 3.9. *Let Assumptions 2.1, 2.2, and 2.3 hold. Let L , a , and $b(a)$ be as in Lemmas 3.6 and 3.8. Suppose that μ in (3.23) satisfies (3.25) for $t > 0$ and (3.34) for $t > T_0 \geq 0$ with $\sigma(T_0) \neq \sigma([T_0]^-)$.*

If $x(0) \in \text{Int}(\overline{\mathcal{E}}_P(R))$, then every trajectory x of the switched system (2.1) with (2.2) satisfies $x(t) \in \text{Int}(\overline{\mathcal{E}}_P(R))$ for $t \geq 0$, and furthermore there exists $T_r \geq 0$ such that $x(t) \in \text{Int}(\underline{\mathcal{E}}_P(ar))$ for $t \geq T_r$.

Remark 3.10. In this section, we have studied the stability analysis of the switched system by the total mismatch time of the modes between the plant and the feedback gain. If the mismatch *does* occurs, the closed-loop system may be *unstable*, If *not*, it is *stable*. Our proposed method is therefore similar to that in [Zhai et al.(2001)], which studied the stability analysis of switched systems with stable and unstable subsystem by the total activation time ratio between stable subsystems and unstable ones. In [Zhai et al.(2001)], the average dwell time introduced by [Hespanha and Morse(1999)] is also required to be sufficiently large. However, such a requirement is not necessary here because we use a common Lyapunov function.

4 Reduction to a Dwell-Time Condition

In the preceding section, we have derived a sufficient condition on the total mismatch time μ for the stabilization of the switched system with limited information. However it may be difficult to check whether μ satisfies (3.25) and (3.34). In this section, we will briefly show that these conditions on μ can be achieved for switching signals with a certain dwell time property.

The proofs of theorems in this section are omitted for space reason.

To proceed, we recall the definition of dwell time. If the switching signal σ has an interval between consecutive discontinuities no smaller than $T_d > 0$, and further if σ has no discontinuities in $[0, T_d)$, then we call σ a *switching signal with dwell time T_d* .

Theorem 4.1. Fix $n \in \mathbb{N}$. For every σ with dwell time nT_s , μ in (3.23) satisfies

$$\mu(t, 0) < \frac{1}{n}t \quad (t > 0).$$

Furthermore, if $\sigma(T_0) \neq \sigma([T_0]^-)$, then

$$\mu(t, T_0) < T_s + \frac{1}{n}(t - T_0) \quad (t > T_0).$$

Theorems 3.9 and 4.1 can be combined in the following way:

Theorem 4.2. Let Assumptions 2.1, 2.2, and 2.3 hold. Let $n \in \mathbb{N}$ satisfy $n \geq 1 + D_P/C_P$. Define

$$a = \exp\left(\frac{T_s(C_P + D_P)}{2}\right),$$

and suppose

$$a \leq \frac{R}{r} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}. \quad (4.1)$$

If $x(0) \in \text{Int}(\bar{\mathcal{E}}_P(R))$ and if σ is a switching signal with dwell time nT_s , then every trajectory $x(t)$ of the switched system (2.1) with (2.2) satisfies $x(t) \in \text{Int}(\bar{\mathcal{E}}_P(R))$ for $t \geq 0$, and furthermore there exists $T_r \geq 0$ such that $x(t) \in \text{Int}(\underline{\mathcal{E}}_P(ar))$ for $t \geq T_r$.

5 Concluding Remarks

We have analyzed the stability of a sampled-data switched systems with a memoryless quantizer. The proposed method uses a common Lyapunov function computed efficiently by a randomized algorithm. The common Lyapunov function leads to the switching conditions on the total mismatch time for quantized state feedback stabilization. We have also examined the relationship between the mismatch time and the dwell time of the switching signal. Future works will focus on stability analysis by multiple Lyapunov functions and an average dwell time property, which can reduce conservativeness in our proposed method.

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